EXTENSIONS OF MULTIPLY TWISTED PLURI-CANONICAL FORMS

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1. Introduction

In this work we study the problem of extending "multiply twisted" pluri-canonical forms from smooth divisors in a complex projective manifold. We first state the main theorem and then review some earlier results. Definitions and notation can be found in Section 2.

Theorem 1.1. *Let X be a projective manifold of dimension n, $D \subset X$ a smooth divisor with canonical section s_D.*

Let h_D be an almost semipositive metric (cf. 2.3) on the line bundle D such that $|s_D|_{h_D}$ is essentially bounded on X, i.e. bounded by a fixed number almost everywhere, and let $(L_1, h_1), \ldots, (L_m, h_m)$ be semipositive line bundles (cf. 2.3) such that the restriction of the singular metric h_j to $L_j|_D$ is well defined, i.e. not identically $+\infty$ along D.

If there is a real number $\mu > 0$ such that

$$\mu\sqrt{-1}\Theta_{h_j} \geqslant \sqrt{-1}\Theta_{h_D}$$

as currents on X for $j = 1, \ldots, m$, then for every section σ of

$$\bigotimes_{j=1}^{m}(K_D + L_j|_D) \otimes \mathscr{I}_1\mathscr{I}_2\cdots\mathscr{I}_m$$

on D, where $\mathscr{I}_j$ denote the multiplier ideal sheaves $\mathscr{I}(h_j|_D)$, there exists a global section $\widetilde{\sigma}$ of

$$\bigotimes_{j=1}^{m}(K_X + D + L_j) = m(K_X + D) + L_1 + \cdots + L_m$$

on X such that $\widetilde{\sigma}|_D = \sigma \wedge (ds_D)^{\otimes m}$ (cf. 2.1).

Extension theorems of this type (for $m = 1$) date back to the work of Ohsawa and Takegoshi [11] on extending holomorphic functions from submanifolds of Stein manifolds with weighted L^2 estimates. Their key idea is to use a modified Bochner–Kodaira inequality to achieve the L^2 estimate for a skewed $\overline{\partial}$ operator. This theorem was generalized by Manivel [10] to the case of holomorphic sections of vector bundles. Variants of their theorems were used by Angehrn and Siu [1], in their study of Fujita's conjecture, to prove the semicontinuity of multiplier ideal sheaves under variation of the singular metrics, and used by Siu [18, 19], in his proof of the invariance of plurigenera, to extend pluricanonical forms from the central fiber of a smooth projective family of complex manifolds to the total space.

The argument exploited in [19] was generally referred to as a "two tower" argument by Siu. Indeed, in [19], the theorem of Ohsawa-Takegoshi type ($m = 1$)

is for the canonical bundle twisted by a suitable line bundle. In passing from a single canonical bundle to pluricanonical bundles, Siu combined the extension theorem with Skoda's theorem on (effective) ideal generation as well as a supremum norm estimate. Later Păun [13] simplified Siu's approach by showing that the supremum norm condition can be replaced by an L^2 one and the invariance of plurigenera can be deduced directly from the extension result without using Skoda's theorem. More precisely, he proved the following result:

Theorem 1.2 (Păun [13]). *Let $\pi : \mathfrak{X} \to \Delta$ be a projective family over the unit disk and (L, h) a semipositive line bundle on $\mathfrak{X}$ such that the restriction $h|_{\mathfrak{X}_0}$ is well defined. Then every section of $(mK_{\mathfrak{X}_0} + L|_{\mathfrak{X}_0}) \otimes \mathscr{I}(h|_{\mathfrak{X}_0})$ on $\mathfrak{X}_0$ extends to a section of $mK_{\mathfrak{X}} + L$.*

His proof consists of an elegant single tower climbing induction argument. The induction is on the multiple of the canonical bundle twisted by the *fixed* line bundle L equipped with a *fixed* singular metric h. It is then natural to ask, when climbing the tower, can we add different line bundles each with its own singular metric instead of just a constant pair (L, h). If this can be achieved, one may possibly obtain an extension theorem of "multiply twisted" pluricanonical forms. In fact, Demailly proved the following result:

Theorem 1.3 (Demailly [3]). *Let $\mathfrak{X}$ and π be as in Theorem 1.2 and (L_j, h_j) $(1 \leqslant j \leqslant m)$ semipositive line bundles on $\mathfrak{X}$ such that $h_j|_{\mathfrak{X}_0}$ are well defined. Suppose $\mathscr{I}(h_j|_{\mathfrak{X}_0}) = \mathscr{O}_{\mathfrak{X}_0}$ for $j = 2, \ldots, m$. Then every section of $(mK_{\mathfrak{X}_0} + L_1|_{\mathfrak{X}_0} + \cdots + L_m|_{\mathfrak{X}_0}) \otimes \mathscr{I}(h_1|_{\mathfrak{X}_0})$ on $\mathfrak{X}_0$ extends to a section of $(mK_{\mathfrak{X}} + L_1 + \cdots + L_m)$.*

Note that, although Theorem 1.3 enables one to add different line bundles L_j, only one of them is allowed to be equipped with a singular metric whose multiplier ideal sheaf is nontrivial. This motivates us to look at the statement like Theorem 1.1, which removes this restriction. This was recently achieved in [20].

Theorem 1.4 ([20]). *Let $\pi : \mathfrak{X} \to \Delta$ be a projective family over the unit disk and (L_j, h_j) $(1 \leqslant j \leqslant m)$ semipositive line bundles on $\mathfrak{X}$ such that $h_j|_{\mathfrak{X}_0}$ are well defined. Then every section of $(mK_{\mathfrak{X}_0} + L_1|_{\mathfrak{X}_0} + \cdots + L_m|_{\mathfrak{X}_0}) \otimes \mathscr{I}_1\mathscr{I}_2 \cdots \mathscr{I}_m$ on $\mathfrak{X}_0$ extends to a section of $(mK_{\mathfrak{X}} + L_1 + \cdots + L_m)$ on $\mathfrak{X}$, where $\mathscr{I}_j$ is the multiplier ideal sheave $\mathscr{I}(h_j|_{\mathfrak{X}_0})$ on $\mathfrak{X}_0$.*

Inspired by the results of Tsuji, Takayama, and Hacon-McKernan respectively in connection with their work on pluricanonical series [15], [14], and [8], we proved our theorem under the setting of pairs of a complex projective manifold and a smooth divisor whose associated line bundle satisfies some conditions on curvature. The projective family case is relatively easier in that the line bundle associated to the central fiber is trivial, hence it can be ignored in the necessary curvature condition, i.e. the curvature inequality in Theorem 1.1 holds automatically.

Most of our arguments in the proof of Theorem 1.1 follow closely Păun's one tower argument. The major new input to overcome the non-triviality of multiplier ideal sheaves $\mathscr{I}(h_j|_D)$, which occurs during the intermediate inductive steps, is a more careful choice of the auxiliary twisting ample line bundle (denoted by A in our argument). This bundle needs to be sufficiently ample to take care of both the required metric properties and the global generation for related coherent sheaves. The complete discussion is presented in Section 3 and Section 4.

For completeness and self-containedness of this article, we include in Appendix 1 (Section 5) a proof of the Ohsawa–Takegoshi type theorem which we will

use. The proof is exactly the same as the proof in [19], except that we deal with the situation in which the line bundle D is not trivial. A similar statement appeared in [17], Theorem 2. It is worth noting that Friedrichs and Hörmander's results ([5] and [7]) on the density in the graph norm (cf. Remark 5.2) plays an essential role when using the Bochner–Kodaira formula to get a priori estimates. This density result requires the weight functions to be smooth or to have at most suitably mild singularities. Therefore, to allow h_D to be a singular metric, one has to reduce the proof to the case when it is smooth. We discuss such a reduction in detail for completeness, although it might be well known to experts. In addition, Theorem 1.1 is a refinement of [17], Theorem 1.

In fact we only dealt with the case h_D being smooth in our first version submitted on October 2010 since we were still struggling on this subtle regularization issue at that time. We developed our treatment in Appendix 1 following ideas of Siu which we learnt from several of his lectures and private notes. We consider a locally biholomorphic projection from a Stein manifold to a Euclidean space and apply the convolution method on the target Euclidean space.

We also noticed that in a recent preprint by Demailly, Hacon, and Păun [4], an extension theorem similar to Theorem 3.1 has been proven. They also gave a detailed discussion on the process of smoothing singular metrics. Their approach is basically as follows. First one imbeds a Stein manifold V (which will be the complement of some suitable sufficiently ample divisor H in the projective manifold X under consideration) in an ambient M (which is an Euclidean space in their case). Then, by a theorem of Siu (Theorem 4.2 in [4]) one can construct a Stein neighborhood W of V in the ambient space M which admits a holomorphic retraction $r : W \to V$. To smoothen plurisubharmonic functions on V, one first pulls them back to W via r, which are still plurisubharmonic. After applying the usual convolution method in the Euclidean space M to regularize the pulled back functions, one takes their restrictions on V.

These two methods are different. Although both methods crucially use the Stein property and convolution, the difference lies in that the approach in [4] is "injective" and ours is "projective".

We are able to extend Theorem 1.1 to allow L_j's to be $\mathbf{R}$ divisors instead of genuine line bundles. We are grateful to the referee for asking this question. Since the proof requires some other techniques, we will present it in a separate work.

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2. Preliminaries and Conventions

2.1. **Adjunction.** Given a smooth divisor D in a compact complex manifold X, we use the same letter D to denote the line bundle associated to D. In order to justify the restriction of sections of adjoint line bundles on X to get sections of adjoint line bundles on D, we need to take a closer look at the adjunction formula $K_D \simeq (K_X + D)|_D$. Locally D is given by a set of equations $\{s_\alpha = 0\}$ with respect to an open cover $\{U_\alpha\}$. The relations $s_\alpha = g_{\alpha\beta} s_\beta$ on $U_\alpha \cap U_\beta$ give a 1-cocycle $\{g_{\alpha\beta}\}$ of the sheave $\mathscr{O}_X^*$ which defines the line bundle D, and tautologically the locally defined functions s_α's give a canonical section, denoted by s_D, which is unique up to scaling and will be fixed throughout all arguments. The short exact sequence

$$0 \to N^*_{D/X} \to T^*_X|_D \to T^*_D \to 0$$

implies a canonical isomorphism by taking wedge product:

$$K_D + N^*_{D/X} = K_X|_D.$$

(We adopt the additive notation for tensor products of line bundles.)

On the other hand, ds_α is a local frame of $N^*_{D/X}$ on U_α. Let e_α be a local frame of D on U_α for all α. The relation $s_\alpha = g_{\alpha\beta} s_\beta$ and $e_\beta = g_{\alpha\beta} e_\alpha$ implies that $\{ds_\alpha \otimes e_\alpha\}$ defines a global frame, denoted by ds_D, of the line bundle $N^*_{D/X} + D|_D$, and hence $N^*_{D/X} + D|_D$ is trivial. This induced the isomorphism

$$K_D \simeq K_D + N^*_{D/X} + D|_D = K_X|_D + D|_D$$

by sending η to $\eta \wedge ds_D$.

2.2. **Singular metrics and pseudonorms.** The term "singular hermitian metric" or "singular metric" always means a hermitian metric whose local weight functions are locally Lebesgue integrable, and hence smooth metrics are counted as singular metrics. For such metrics h we use Θ_h to denote their curvature currents. Locally we have $h = e^{-\varphi}$ with $\Theta_h = -\partial\bar{\partial} \log e^{-\varphi} = \partial\bar{\partial}\varphi$.

Let X be a complex manifold of dimension n and L a line bundle on X with a singular metric h. Let s be a (Lebesgue) measurable section of $mK_X + L$. Suppose s and h are represented by functions $f(z)$ and $h(z)$ in terms of local coordinates $z = (z^1, \ldots, z^n)$, $z^j = x^j + \sqrt{-1} y^j$, of trivializing charts of L.

Definition 2.1. We define a measurable (n,n)-form $\langle s \rangle_h^{\frac{2}{m}}$ by setting

$$\langle s \rangle_h^{\frac{2}{m}} = h(z)^{\frac{1}{m}} |f(z)|^{\frac{2}{m}} dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n$$

locally. $\langle s \rangle_h^{\frac{2}{m}}$ is clearly well defined and is nonnegative with respect to the canonical orientation on X associated to $dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n$. Therefore we define

$$\langle\langle s \rangle\rangle_h = \int_X \langle s \rangle_h^{\frac{2}{m}} \leqslant \infty.$$

This number is called the pseudonorm of s with respect to h.

Suppose g is a smooth hermitian metric on T_X with Kähler form ω. g induces a hermitian metric on the canonical bundle K_X, denoted as g_ω. Let $dV_\omega = \dfrac{\omega^n}{n!}$ be the volume form on X induced by g. It is easily seen that

$$\langle s \rangle_h^{\frac{2}{m}} = |s|^{\frac{2}{m}}_{g_\omega^{\otimes m} \otimes h} dV_\omega.$$

Using this expression one sees directly the following facts:

(i) Suppose L and L' are two line bundles with singular metrics h and h' respectively. For any measurable sections s of $mK_X + L$ and s' of L', and $l \in \mathbf{N}$ we have

$$\langle s \otimes s' \rangle_{h\otimes h'}^{\frac{2}{m}} = |s'|_{h'}^{\frac{2}{m}} \langle s \rangle_h^{\frac{2}{m}} \tag{2.1}$$

and

$$\langle s^l \rangle_{h^{\otimes l}}^{\frac{2}{lm}} = \langle s \rangle_h^{\frac{2}{m}}. \tag{2.2}$$

(ii) If s_j is a measurable section of $m_j K_X + L_j$ and h_j is a singular metric on L_j, $j = 1, \ldots, r$, then we can deduce from the usual Hölder inequality the "Hölder inequality for pseudonorms":

$$\langle\langle s_1 \otimes \cdots \otimes s_r \rangle\rangle_{h_1\otimes\cdots\otimes h_r}^{m_1+\cdots+m_r} \leqslant \langle\langle s_1 \rangle\rangle_{h_1}^{m_1} \cdots \langle\langle s_r \rangle\rangle_{h_r}^{m_r}. \tag{2.3}$$

2.3. **Almost semipositive line bundles and pseudoeffective divisors.** A *semipositive* line bundle (resp. an almost semipositive line bundle) is a pair (L, h) of a line bundle L and a singular hermitian metric h on L such that $\sqrt{-1}\Theta_h$ is a *closed positive current* in the sense of Lelong (resp. the sum of a closed positive current and a smooth $(1,1)$-form), or equivalently, each of its local weights is a nontrivial plurisubharmonic function, i.e. not identically $-\infty$ (resp. the sum of a nontrivial plurisubharmonic function and a smooth function). We will call such h a semipositive metric (resp. an almost semipositive metric) on L. The multiplier ideal sheaf associated to an almost semipositive singular metric h is the coherent sheaf of local L_h^2 sections and is denoted by $\mathscr{I}_h$ or by $\mathscr{I}(h)$.

Remark 2.1. On a projective manifold X, a pair (L, h) is almost semipositive if and only if there exist a semipositive line bundle (L_1, h_1) and a line bundle with smooth hermitian metric (L_2, h_2) such that $L = L_1 \otimes L_2$ and $h = h_1 \otimes h_2$.

A typical type of semipositive line bundles consists of effective line bundles by the following construction.

Definition 2.2. Let $S = \{s_1, \ldots, s_l\}$ be a set of nontrivial global holomorphic sections of a line bundle L. For any $\sigma \in L_x$ where $x \in X$, we choose an arbitrary smooth metric h on L and define

$$|\sigma|_{h_S}^2 := \frac{|\sigma|_h^2}{\sum_{j=1}^{l} |s_j(x)|_h^2}.$$

If s is a section of $K_X + L$ and $S = \{s_1, \ldots, s_l\}$ a set of global holomorphic section of L, then for any smooth metric h on L we have

$$\langle s \rangle_{h_S}^2 = \frac{\langle s \rangle_h^2}{\sum_{j=1}^{l} |s_j|_h^2}. \tag{2.4}$$

It is clear that the definition does not depend on the choice of h. Locally if the sections $\{s_j\}$ are represented by functions $\{f_j\}$ then the weight function is

$$\varphi := \log\Big(\sum_{j=1}^{l} |f_j|^2\Big)$$

which is plurisubharmonic, and hence $\sqrt{-1}\Theta_{h_S} = \sqrt{-1}\partial\bar\partial\log(\Sigma_j|f_j|^2) \geqslant 0$.

Denote by $\mathrm{Psef}(X) \subseteq N^1(X)_{\mathbf{R}}$ the closure of the real convex cone generated by numerical classes of semipositive line bundles over X. In the algebraic case, we have the following interpretation.

Remark 2.2. (cf. [2]) If X is projective then $\mathrm{Psef}(X) = \overline{\mathrm{Eff}(X)} = \overline{\mathrm{Big}(X)}$, where $\overline{\mathrm{Eff}(X)}$ (resp. $\overline{\mathrm{Big}(X)}$) is the closure of effective (resp. big) cone of X, which is also known as the cone of pseudoeffective divisors.

3. The main extension result

3.1. **An extension theorem for adjoint line bundles.** We will need the following extension theorem of Ohsawa–Takegoshi type for adjoint line bundles, whose proof will be given in Appendix 1.

Theorem 3.1. *Let X be a projective manifold, $D \subseteq X$ a smooth divisor. Suppose h_D an almost semipositive metric on the line bundle D such that $|s_D|_{h_D}$ is essentially bounded on X and (L, h) be a semipositive line bundle on X. If there is a real number $\mu > 0$ such that*

$$\mu\sqrt{-1}\Theta_h \geqslant \sqrt{-1}\Theta_{h_D}$$

as currents on X, then for every section s of $(K_D + L|_D) \otimes \mathscr{I}(h|_D)$ there exists a section $\widetilde{s}$ of $K_X + D + L$ such that $\widetilde{s}|_D = s \wedge ds_D$ and

$$\int_X \langle\widetilde{s}\rangle^2_{h_D\otimes h} \leqslant C\int_D \langle s\rangle^2_h$$

where $C > 0$ only depends on $\operatorname{ess.sup}_X|s_D|_{h_D}$ *and μ.*

Note that the statement of Theorem 1.1 for $m = 1$ is exactly the statement of Theorem 3.1. Hence we fix from now on a positive integer $m \geq 2$ and consider a non-zero σ as in the hypothesis of Theorem 1.1.

3.2. **Reduction to constructing a semipositive metric on** $m(K_X + D) + \sum_1^m L_j$**.** Note that $m(K_X + D) + \sum_1^m L_j = K_X + D + (m-1)(K_X + D) + \sum_1^m L_j$. In order to prove Theorem 1.1 via Theorem 3.1, we need to create a semipositive metric h_0 on $(m-1)(K_X + D) + \sum_1^m L_j$ such that

$$\mu\sqrt{-1}\Theta_{h_0} \geqslant \sqrt{-1}\Theta_{h_D}$$

as currents and

$$\int_D \langle\sigma \wedge ds_D^{\otimes(m-1)}\rangle^2_{h_0} < \infty.$$

The construction of h_0 goes as follows. First, we choose A to be so ample that the following conditions hold:

(A_1) For each $r = 0, 1, \ldots, m-1$, the line bundle $(m-r)A$ is generated by its global sections $\{t_l^{(r)}\}_{1 \leqslant l \leqslant N}$.

(A_2) The coherent sheaf $(K_D + L_j|_D + A|_D) \otimes \mathscr{I}_j$ on D is generated by its global sections $\{s_{j,l}\}_{1 \leqslant l \leqslant N}$ for each $1 \leqslant j \leqslant m$.

(A_3) The following map induced by $\mathscr{I}_1 \otimes \cdots \otimes \mathscr{I}_m \to \mathscr{I}_1 \cdots \mathscr{I}_m$ is surjective:

$$\bigotimes_{j=1}^{m} H^0\left(D, (K_D + L_j|_D + A|_D) \otimes \mathscr{I}_j\right) \longrightarrow H^0\left(D, (mK_D + \textstyle\sum_1^m L_j|_D + mA|_D) \otimes \mathscr{I}_1 \cdots \mathscr{I}_m\right).$$

This can be achieved by Lemma 6.1 in Appendix 2.

(A_4) Every section of $\left(m(K_X + D) + \sum_1^m L_j + mA\right)|_D$ on D extends to X. This is a consequence of the Serre vanishing theorem.

Suppose that we have a semipositive metric h_∞ (which will be constructed in Lemma 4.3 by using the auxiliary ample bundle A) on $m(K_X + D) + \sum_1^m L_j$ such that $\left|\sigma \wedge ds_D^{\otimes m}\right|_{h_\infty} \leqslant 1$. We take $h_0 = h_\infty^{\frac{m-1}{m}} (h_1 \cdots h_m)^{\frac{1}{m}}$. The curvature condition holds since

$$\mu\sqrt{-1}\Theta_{h_0} = \frac{\mu(m-1)}{m}\sqrt{-1}\Theta_{h_\infty} + \frac{1}{m}\sum_{j=1}^{m} \mu\sqrt{-1}\Theta_{h_j} \geqslant \sqrt{-1}\Theta_{h_D}$$

by the curvature assumption in Theorem 1.1.

The finiteness condition also holds. To see this, first note that, by (2.1) and (2.2),

$$\begin{aligned}
\langle \sigma \wedge ds_D^{\otimes(m-1)} \rangle_{h_0}^2 &= \langle (\sigma \wedge ds_D^{\otimes(m-1)})^{\otimes m} \rangle_{h_0^{\otimes m}}^{\frac{2}{m}} \\
&= \langle (\sigma \wedge ds_D^{\otimes m})^{\otimes(m-1)} \otimes \sigma \rangle_{h_\infty^{\otimes(m-1)} \otimes h_1 \otimes \cdots \otimes h_m}^{\frac{2}{m}} \\
&= \left|(\sigma \wedge ds_D^{\otimes m})^{\otimes(m-1)}\right|_{h_\infty^{\otimes(m-1)}}^{\frac{2}{m}} \langle \sigma \rangle_{h_1 \otimes \cdots \otimes h_m}^{\frac{2}{m}} \\
&= \left(\left|\sigma \wedge ds_D^{\otimes m}\right|_{h_\infty}^2\right)^{\frac{m-1}{m}} \langle \sigma \rangle_{h_1 \otimes \cdots \otimes h_m}^{\frac{2}{m}} \leqslant \langle \sigma \rangle_{h_1 \otimes \cdots \otimes h_m}^{\frac{2}{m}}.
\end{aligned}$$

By (A_3),

$$\sigma \otimes t_l^{(0)} = \sum_{p=1}^{n_l} \tau_{l;1,p} \otimes \cdots \otimes \tau_{l;m,p}$$

where $\tau_{l;j,p}$ are sections of $(K_D + L_j|_D + A|_D) \otimes \mathscr{I}_{h_j|D}$ for $l = 1, \ldots, N$. Again, by (2.1) and (2.2),

$$\Big(\sum_{l=1}^{N} \big|t_l^{(0)}\big|^{\frac{2}{m}}_{h_A^{\otimes m}}\Big) \langle\sigma\rangle^{\frac{2}{m}}_{h_1\otimes\cdots\otimes h_m} = \sum_{l=1}^{N} \langle \sigma \otimes t_l^{(0)}\rangle^{\frac{2}{m}}_{h_1\otimes\cdots\otimes h_m\otimes h_A^{\otimes m}}$$
$$\leqslant \sum_{l=1}^{N}\sum_{p=1}^{n_l} \langle \tau_{l;1,p} \otimes \cdots \otimes \tau_{l;m,p}\rangle^{\frac{2}{m}}_{h_1\otimes\cdots\otimes h_m\otimes h_A^{\otimes m}}$$

where h_A is a fixed smooth metric on A.

$$M_0 := \min_D \sum_l \big|t_l^{(0)}\big|^{\frac{2}{m}}_{h_A^{\otimes m}} > 0$$

exists since $\sum\limits_l \big|t_l^{(0)}\big|^{\frac{2}{m}}_{h_A^{\otimes m}}$ is a nonvanishing smooth function by (A_1) and D is compact. Therefore

$$\langle\sigma\rangle^{\frac{2}{m}}_{h_1\otimes\cdots\otimes h_m} \leqslant \frac{1}{M_0}\sum_{l=1}^{N}\sum_{p=1}^{n_l} \langle \tau_{l;1,p} \otimes \cdots \otimes \tau_{l;m,p}\rangle^{\frac{2}{m}}_{h_1\otimes\cdots\otimes h_m\otimes h_A^{\otimes m}}.$$

By the above, (A_1), and (2.3),

$$\begin{aligned}
&\int_D \langle \sigma \wedge ds_D^{\otimes(m-1)}\rangle^2_{h_0} \\
&\leqslant \frac{1}{M_0}\sum_{l=1}^{N}\sum_{p=1}^{n_l} \int_D \langle \tau_{l;1,p} \otimes \cdots \otimes \tau_{l;m,p}\rangle^{\frac{2}{m}}_{h_1\otimes\cdots\otimes h_m\otimes h_A^{\otimes m}} \\
&\leqslant \frac{1}{M_0}\sum_{l=1}^{N}\sum_{p=1}^{n_l} \Big(\int_D \langle\tau_{l;1,p}\rangle^2_{h_1\otimes h_A}\Big)^{\frac{1}{m}} \cdots \Big(\int_D \langle\tau_{l;m,p}\rangle^2_{h_m\otimes h_A}\Big)^{\frac{1}{m}} < \infty.
\end{aligned}$$

Applying Theorem 3.1 to prove Theorem 1.1 is then justified if such h_∞ exists.

4. CONSTRUCTION OF THE METRIC h_∞

4.1. **A modification of Siu and Păun's induction.** Here we follow the argument in [13] and [19]. For every positive integer $k = qm + r$ ($q = [k/m]$ the Gauss symbol of k/m and $0 \leqslant r \leqslant m-1$ the remainder), we let

$$L^{(k)} := q\sum_{j=1}^{m} L_j + L_1 + \cdots + L_r$$

and let $F_k := k(K_X + D) + L^{(k)} + mA$ where A is the ample bundle chosen in 3.2.

Were $m(K_X + D) + L^{(m)}$ known to have a family of sections which do not vanish identically along D and their restrictions to D are basically $\sigma \wedge ds_D^{\otimes(m)}$ multiplied by some functions which do not have common zeros, we can simply take h_∞ to be the semipositive metric defined by them (Definition 2.2).

However, we do not know a priori that $m(K_X + D) + L^{(m)}$ have any nonzero sections (we are trying to produce one). Instead, for the ample line bundle A we can find a set of sections S_k of $F_k = k(K_X + D) + L^{(k)} + mA$ whose restrictions to D have properties similar to those mentioned above (Lemma 4.1). Then we

try to obtain h_∞ by "taking the q-root" of the semipositive metrics $h_{S_{qm}}$ on $F_{qm} = q(mK_X + mD + L^{(m)}) + mA$ to "eliminate" the line bundle factor mA (Lemma 4.3).

Now we let $\Lambda_r := \prod_1^r \{1, \ldots, N\}$ for $r = 1, 2, \ldots, m-1$. For every $J = (j_1, \ldots, j_r) \in \Lambda_r$, we define

$$s_J^{(r)} := s_{1,j_1} \otimes \cdots \otimes s_{r,j_r}$$

with the convention that $\Lambda_0 = \{0\}$ $s_0^{(0)} := 1$ for $r = 0$. We define the special index set Λ_m^* to be $\prod_1^m \{1, \ldots, N\}$ and sections $\hat{s}_J^{(m)} = s_{1,j_1} \otimes \cdots \otimes s_{m,j_m}$ for all $J = (j_1, \ldots, j_m) \in \Lambda_m^*$. We consider for each $k \geq m$ the following statement:

$(E)_k$: There exists a family of sections

$$S_k = \{\widetilde{\sigma}_{J,l}^{(k)} : J \in \Lambda_r, 1 \leqslant l \leqslant N\}$$

of F_k over X such that

$$\widetilde{\sigma}_{J,l}^{(k)}|_D = \sigma^{\otimes [k/m]} \otimes s_J^{(r)} \otimes t_l^{(r)} \wedge ds_D^{\otimes k} \tag{4.1}$$

for all $J \in \Lambda_r$ and $l = 1, \ldots, N$, where $r = k - [k/m]m$.

Lemma 4.1. *$(E)_k$ holds for all $k \geqslant m$. Moreover, there exists a constant $C_0 > 0$ which only depends on* $\operatorname{ess.sup}_X |s_D|_{h_D}$*, μ, σ, and the choices of $\{t_l^{(r)}\}$ and $\{s_{j,l}\}$ in (A_2) and (A_3) above such that*

$$\int_X \sum_{\substack{J \in \Lambda_r \\ l=1,\ldots,N}} \langle \widetilde{\sigma}_{J,l}^{(k)} \rangle^2_{h_D \otimes h_{S_{k-1}} \otimes h_{r^*}} \leqslant C_0 \tag{4.2}$$

for all $k > m$, where $r = k - [k/m]m$ and

$$r^* := \begin{cases} r & \text{if} \quad r \neq 0, \\ m & \text{if} \quad r = 0. \end{cases}$$

Proof. First, $(E)_m$ holds by (A_4). We proceed to prove that $(E)_{k-1}$ implies $(E)_k$ for any $k > m$. Note that $F_k = K_X + D + F_{k-1} + L_{r^*}$ and hence $F_k|_D = K_D + (F_{k-1} + L_{r^*})|_D + (N_{D/X}^* + D|_D)$ by 2.1. We are going to apply Theorem 3.1 to the situation $L = F_{k-1} + L_{r^*}$ and $s = \sigma^{\otimes [k/m]} \otimes s_J^{(r)} \otimes t_l^{(r)} \wedge ds_D^{\otimes (k-1)}$. We choose the singular metric h on $F_{k-1} + L_{r^*}$ to be $h_{S_{k-1}} \otimes h_{r^*}$.

The restriction $h_{S_{k-1}}|_D$ is well defined by (4.1), (A_1), (A_2), and (A_3); $h_{r^*}|_D$ is well defined by the hypothesis of Theorem 1.1. Therefore $h|_D$ is well defined. By 2.3 and the hypothesis of Theorem 1.1,

$$\mu\sqrt{-1}\Theta_h = \mu\sqrt{-1}\Theta_{h_{S_{k-1}}} + \mu\sqrt{-1}\Theta_{h_{r^*}} \geqslant \sqrt{-1}\Theta_{h_D}$$

and the curvature condition is fulfilled.

In the following we will show that

$$\int_D \langle \sigma^{\otimes [k/m]} \otimes s_J^{(r)} \otimes t_l^{(r)} \wedge ds_D^{\otimes (k-1)} \rangle^2_{h_{S_{k-1}} \otimes h_{r^*}} \leqslant C'$$

for a positive number C' which only depends on the choices of $\{t_l^{(r)}\}$ and $\{s_{j,l}\}$ in (A_2) and (A_3) above. This will imply s is a section of $\big(K_D + (F_{k-1} + L_{r^*})|_D\big) \otimes \mathscr{I}_h$ and, combined with the pseudonorm inequality on Theorem 3.1, will yield (4.2).

Case 1: $r \neq 0$, i.e. $[k/m] = [(k-1)/m]$.

We choose smooth metrics h_A on $A|_D$, $h^{(r-1)}$ on $(r-1)K_D + L^{(r-1)}|_D$, and h' on $[k/m](mK_D + L^{(m)}) + (k-1)(N^*_{D/X} + D|_D)$. We let $h := h' \otimes h^{(r-1)} \otimes h_A^{\otimes m}$ on $F_{k-1}|_D$. Writing $J = (J'_0, j_0)$ with $J'_0 \in \Lambda_{r-1}$, by (2.1), (2.4), and (4.1), we have

$$\begin{aligned}
&\langle \sigma^{\otimes [k/m]} \otimes s_J^{(r)} \otimes t_l^{(r)} \wedge ds_D^{\otimes (k-1)} \rangle^2_{h_{S_{k-1}} \otimes h_r} \\
&\quad = \frac{\langle \sigma^{\otimes [k/m]} \otimes s_J^{(r)} \otimes t_l^{(r)} \wedge ds_D^{\otimes (k-1)} \rangle^2_{h \otimes h_r}}{\sum\limits_{\substack{J' \in \Lambda_{r-1} \\ l'=1,\dots,N}} \left| \sigma^{\otimes [(k-1)/m]} \otimes s_{J'}^{(r-1)} \otimes t_{l'}^{(r-1)} \wedge ds_D^{\otimes (k-1)} \right|^2_h} \\
&\quad = \frac{\left| \sigma^{\otimes [k/m]} \wedge ds_D^{\otimes (k-1)} \right|^2_{h'} \left| s_{J'_0}^{(r-1)} \right|^2_{h^{(r-1)}} \langle s_{r,j_0} \rangle^2_{h_A \otimes h_r} \left| t_l^{(r)} \right|^2_{h_A^{\otimes (m-r)}}}{\sum\limits_{\substack{J' \in \Lambda_{r-1} \\ l'=1,\dots,N}} \left| \sigma^{\otimes [k/m]} \wedge ds_D^{\otimes (k-1)} \right|^2_{h'} \left| s_{J'}^{(r-1)} \right|^2_{h^{(r-1)}} \left| t_{l'}^{(r-1)} \right|^2_{h_A^{\otimes (m-r+1)}}} \\
&\quad = \frac{\left| s_{J'_0}^{(r-1)} \right|^2_{h^{(r-1)}}}{\sum\limits_{J' \in \Lambda_{r-1}} \left| s_{J'}^{(r-1)} \right|^2_{h^{(r-1)}}} \times \frac{\left| t_l^{(r)} \right|^2_{h_A^{\otimes (m-r)}}}{\sum\limits_{l'=1}^{N} \left| t_{l'}^{(r-1)} \right|^2_{h_A^{\otimes (m-r+1)}}} \langle s_{r,j_0} \rangle^2_{h_A \otimes h_r} \\
&\quad \leqslant \frac{\left| t_l^{(r)} \right|^2_{h_A^{\otimes (m-r)}} \langle s_{r,j_0} \rangle^2_{h_A \otimes h_r}}{\sum\limits_{l'=1}^{N} \left| t_{l'}^{(r-1)} \right|^2_{h_A^{\otimes (m-r+1)}}}.
\end{aligned}$$

By (A_1) and the choices of $s_{r,j}$,

$$C_1 := \max_{l,r} \int_D \frac{\left| t_l^{(r)} \right|^2_{h_A^{\otimes (m-r)}} \langle s_{r,j} \rangle^2_{h_A \otimes h_r}}{\sum\limits_{l'=1}^{N} \left| t_{l'}^{(r-1)} \right|^2_{h_A^{\otimes (m-r+1)}}}$$

exists. It is clear that

$$\int_D \langle \sigma^{\otimes [k/m]} \otimes s_J^{(r)} \otimes t_l^{(r)} \wedge ds_D^{\otimes (k-1)} \rangle^2_{h_{S_{k-1}} \otimes h_r} \leqslant C_1.$$

Case 2: $r = 0$, i.e. $[k/m] = [(k-1)/m] + 1$.

We choose smooth metrics h_A on $A|_D$, $h^{(m-1)}$ on $(m-1)K_D + L^{(m-1)}|_D$, $\hat{h}$ on $K_D + L_m|_D + A|_D$, and h' on $[(k-1)/m](mK_D + L^{(m)}) + (k-1)(N^*_{D/X} + D|_D)$. We let $h := h' \otimes h^{(m-1)} \otimes h_A^{\otimes m}$ on $F_{k-1}|_D$. Now $J \in \Lambda_0 = \{0\}$, by (2.1), (2.4), and (4.1), we have

$$
\begin{aligned}
&\langle \sigma^{\otimes[k/m]} \otimes s_0^{(0)} \otimes t_l^{(0)} \wedge ds_D^{\otimes(k-1)} \rangle^2_{h_{S_{k-1}} \otimes h_m} \\
&\quad = \frac{\langle \sigma^{\otimes[k/m]} \otimes t_l^{(0)} \wedge ds_D^{\otimes(k-1)} \rangle^2_{h \otimes h_m}}{\sum\limits_{\substack{J' \in \Lambda_{m-1} \\ l'=1,\dots,N}} \left| \sigma^{\otimes[(k-1)/m]} \otimes s_{J'}^{(m-1)} \otimes t_{l'}^{(m-1)} \wedge ds_D^{\otimes(k-1)} \right|^2_h} \\
&\quad = \frac{\left| \sigma^{\otimes[(k-1)/m]} \wedge ds_D^{\otimes(k-1)} \right|^2_{h'} \langle \sigma \otimes t_l^{(0)} \rangle^2_{h^{(m-1)} \otimes h_A^{\otimes m} \otimes h_m}}{\sum\limits_{\substack{J' \in \Lambda_{m-1} \\ l'=1,\dots,N}} \left| \sigma^{\otimes[(k-1)/m]} \wedge ds_D^{\otimes(k-1)} \right|^2_{h'} \left| s_{J'}^{(m-1)} \right|^2_{h^{(m-1)} \otimes h_A^{\otimes(m-1)}} \left| t_{l'}^{(m-1)} \right|^2_{h_A}} \\
&\quad = \frac{\langle \sigma \otimes t_l^{(0)} \rangle^2_{h^{(m-1)} \otimes h_A^{\otimes m} \otimes h_m}}{\sum\limits_{J' \in \Lambda_{m-1}} \left| s_{J'}^{(m-1)} \right|^2_{h^{(m-1)} \otimes h_A^{\otimes(m-1)}} \sum\limits_{l'=1}^{N} \left| t_{l'}^{(m-1)} \right|^2_{h_A}}.
\end{aligned}
$$

By multiplying both the numerator and the denominator by the same positive factor $\sum\limits_{j=1}^{N} |s_{m,j}|^2_{\hat{h}}$, the expression becomes

$$
\begin{aligned}
&\frac{\sum\limits_{j=1}^{N} \left| s_{m,j} \right|^2_{\hat{h}} \langle \sigma \otimes t_l^{(0)} \rangle^2_{h^{(m-1)} \otimes h_A^{\otimes m} \otimes h_m}}{\sum\limits_{\substack{J' \in \Lambda_{m-1} \\ j=1,\dots,N}} \left| s_{J'}^{(m-1)} \right|^2_{h^{(m-1)} \otimes h_A^{\otimes(m-1)}} \left| s_{m,j} \right|^2_{\hat{h}} \sum\limits_{l'=1}^{N} \left| t_{l'}^{(m-1)} \right|^2_{h_A}} \\
&\quad = \frac{\sum\limits_{j=1}^{N} \langle \sigma \otimes t_l^{(0)} \otimes s_{m,j} \rangle^2_{h^{(m-1)} \otimes \hat{h} \otimes h_A^{\otimes m} \otimes h_m}}{\sum\limits_{J \in \Lambda_m^*} \left| \hat{s}_J^{(m)} \right|^2_{h^{(m-1)} \otimes \hat{h} \otimes h_A^{\otimes(m-1)}} \sum\limits_{l'=1}^{N} \left| t_{l'}^{(m-1)} \right|^2_{h_A}} \\
&\quad = \frac{\left| \sigma \otimes t_l^{(0)} \right|^2_{h^{(m-1)} \otimes \hat{h} \otimes h_A^{\otimes(m-1)}}}{\sum\limits_{J \in \Lambda_m^*} \left| \hat{s}_J^{(m)} \right|^2_{h^{(m-1)} \otimes \hat{h} \otimes h_A^{\otimes(m-1)}}} \times \frac{\sum\limits_{j=1}^{N} \langle s_{m,j} \rangle^2_{h_A \otimes h_m}}{\sum\limits_{l'=1}^{N} \left| t_{l'}^{(m-1)} \right|^2_{h_A}}.
\end{aligned}
$$

By (A_1) and the choices of $s_{m,j}$,

$$
C_2 := \sum_{j=1}^{N} \int_D \frac{\langle s_{m,j} \rangle^2_{h_A \otimes h_m}}{\sum\limits_{l'=1}^{N} \left| t_{l'}^{(m-1)} \right|^2_{h_A}}
$$

exists. By (A_2) and (A_3), $\sigma \otimes t_l^{(0)}$ is a linear combination of $\{\hat{s}_J^{(m)}\}_{J\in\Lambda_m^*}$. The Cauchy–Schwartz inequality implies that

$$C_3 := \max_{l=1,\dots,N} \sup_D \frac{\left|\sigma \otimes t_l^{(0)}\right|^2_{h^{(m-1)}\otimes\hat{h}\otimes h_A^{\otimes(m-1)}}}{\sum_{J\in\Lambda_m^*} \left|\hat{s}_J^{(m)}\right|^2_{h^{(m-1)}\otimes\hat{h}\otimes h_A^{\otimes(m-1)}}}$$

exists. In this case we have

$$\int_D \langle \sigma^{\otimes[k/m]} \otimes s_0^{(0)} \otimes t_l^{(0)} \wedge ds_D^{\otimes(k-1)}\rangle^2_{h_{S_{k-1}}\otimes h_m} \leqslant C_2C_3.$$

It is clear that C_1, C_2, C_3, and hence

$$C' := \max\{C_1, C_2C_3\}$$

depend only on σ and the choices of $\{t_l^{(r)}\}$ and $\{s_{j,l}\}$ in (A_2) and (A_3) (and is independent of $k \geqslant m$ and the choices of the auxiliary metrics h_A, $h^{(m-1)}$ and $\hat{h}$).

In summary,

$$\int_D \langle \sigma^{\otimes[k/m]} \otimes s_J^{(r)} \otimes t_l^{(r)} \wedge ds_D^{\otimes(k-1)}\rangle^2_{h_{S_{k-1}}\otimes h_{r^*}} \leqslant C'.$$

By Theorem 3.1, there exists a family of sections

$$S_k = \{\widetilde{\sigma}_{J,l}^{(k)} : J \in \Lambda_k, 1 \leqslant l \leqslant N\}$$

of F_k over X such that

$$\widetilde{\sigma}_{J,l}^{(k)}|_D = \sigma^{\otimes[k/m]} \otimes s_J^{(r)} \otimes t_l^{(r)} \wedge ds_D^{\otimes k}$$

and

$$\int_X \sum_{\substack{J\in\Lambda_r\\ l=1,\dots,N}} \langle \widetilde{\sigma}_{J,l}^{(k)}\rangle^2_{h_D\otimes h_{S_{k-1}}\otimes h_r} \leqslant C_0 := CC'.$$

This completes the proof. □

4.2. **Siu's construction of the metric** h_∞. For any $w_0 = (w_0^1, \dots, w_0^n) \in \mathbf{C}^n$ and any $r > 0$, we let $D_r(w_0)$ denote $\{(w^1, \dots, w^n) : |w^\nu - w_0^\nu| < r, 1 \leqslant \nu \leqslant n\}$, the polydisk in $\mathbf{C}^n$ centered at w_0 with polyradii $(r, \dots, r)$. Choose a finite open cover $\mathcal{W}' = \{W'_\alpha\}_{\alpha\in I}$ of X such that each W'_α is biholomorphic to $D_1(0)$ and $\mathcal{W} = \{W_\alpha\}$ also covers X, where $W_\alpha \subseteq W'_\alpha$ corresponds to $D_{1/3}(0)$. We also require that $L_1|_{W'_\alpha}, \dots, L_m|_{W'_\alpha}$, $D|_{W'_\alpha}$, and $A|_{W'_\alpha}$ (and hence $F_k|_{W'_\alpha}$, $k \geqslant m$) are trivial for all $\alpha \in I$.

Suppose that $\widetilde{\sigma}_{J,l}^{(k)}$ given by Lemma 4.1 are represented by holomorphic functions $\widetilde{f}_{\alpha;J,l}^{(k)}$ on W'_α for each $\alpha \in I$.

Lemma 4.2. *There exists $C'_0 > 0$ such that*

$$\max_{\substack{x\in W_\alpha\\ \alpha\in I}} \frac{1}{q} \log \sum_{l=1}^N \left|\widetilde{f}_{\alpha;0,l}^{(qm)}(x)\right|^2 \le C'_0$$

for all $q \in \mathbf{N}$.

The essential part of this result is the uniformity of C'_0 with respect to $q \in \mathbf{N}$.

Proof. For each $x \in W_\alpha$ whose coordinate is $w_x = (w_x^1, \ldots, w_x^n)$, we let W_x be the subset of W'_α corresponding to $D_{1/3}(w_x)$. Since $\bigcup_{x \in W'_\alpha} W_x \Subset W'_\alpha$, there exists $M > 0$ such that on all W_x we have (following the notation in 4.1)

$$\frac{\sum_{J,l} \left|f^{(k+1)}_{\alpha;,l}\right|^2}{\sum_{J',l'} \left|f^{(k)}_{\alpha;J',l'}\right|^2} dV \leq M \sum_{J,l} \langle \widetilde{\sigma}^{(k+1)}_{\alpha;J,l} \rangle^2_{h_D \otimes h_{S_{k-1}} \otimes h_{r^*}} \tag{4.3}$$

where $u^\nu = \mathrm{Re}\ w^\nu$, $v^\nu = \mathrm{Im}\ w^\nu$, and $dV = du^1 \wedge dv^1 \wedge \cdots \wedge du^n \wedge dv^n$. For each $m \leqslant k \leqslant qm - 1$, by Jensen's inequality, (4.3), and Lemma 4.1,

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}\big(D_{1/3}(w_x)\big)} \int_{D_{1/3}(w_x)} \log \sum_{J,l} \left|f^{(k+1)}_{\alpha;J,l}\right|^2 dV \\
&\qquad - \frac{1}{\mathrm{Vol}\big(D_{1/3}(w_x)\big)} \int_{D_{1/3}(w_x)} \log \sum_{J',l'} \left|f^{(k)}_{\alpha;J',l'}\right|^2 dV \\
&\leqslant \log \left(\frac{1}{\mathrm{Vol}\big(D_{1/3}(w_x)\big)} \int_{D_{1/3}(w_x)} \frac{\sum_{J,l} \left|f^{(k+1)}_{\alpha;J,l}\right|^2}{\sum_{J',l'} \left|f^{(k)}_{\alpha;J',l'}\right|^2} dV \right) \\
&\leqslant \log \left(\frac{1}{\mathrm{Vol}(W_x)} \int_{W_x} \sum_{J,l} \langle \widetilde{\sigma}^{(k+1)}_{\alpha;J,l} \rangle^2_{h_D \otimes h_{S_{k-1}} \otimes h_{r^*}} \right) \\
&\leqslant \log \frac{MC_0}{\mathrm{Vol}(W_x)}.
\end{aligned}$$

Summing up the above computation from $k = m$ to $k = qm - 1$ and applying the sub-mean value inequality, we obtain

$$\begin{aligned}
&\frac{1}{q} \log \sum_{l=1}^N \left|\widetilde{f}^{(qm)}_{\alpha;0,l}(x)\right|^2 \\
&\leqslant \frac{1}{q\mathrm{Vol}\big(D_{1/3}(w_x)\big)} \int_{D_{1/3}(w_x)} \log \sum_{l=1}^N \left|f^{(qm)}_{\alpha;0,l}\right|^2 dV \\
&\leqslant \frac{9^n}{q\pi^n} \int_{D_{1/3}(w_x)} \log \sum_{l=1}^N \left|f^{(m)}_{\alpha;0,l}\right|^2 dV + \frac{(q-1)m}{q} \log \frac{9^n MC_0}{\pi^n} \\
&\leqslant \frac{9^n}{q\pi^n} \int_{D_{2/3}(0)} \log \sum_{l=1}^N \left|f^{(m)}_{\alpha;0,l}\right|^2 dV + \frac{(q-1)m}{q} \log \frac{9^n MC_0}{\pi^n}.
\end{aligned}$$

Since we have only finitely many $\alpha \in I$, the expected constant $C'_0 > 0$ clearly exists. □

Now we are ready to construct the desired metric h_∞.

Lemma 4.3. *There exists a semipositive metric h_∞ on $m(K_X + D) + L^{(m)}$ such that $|\sigma \wedge ds_D^{\otimes m}|_{h_\infty} \leqslant 1$.*

Proof. On each $W_\alpha \in \mathcal{W}$ we let

$$\widetilde{f}_\alpha^{(\infty)} := \lim_{p\to\infty}\left(\sup_{q\geqslant p}\frac{1}{q}\log\sum_{l=1}^{N}|\widetilde{f}_{\alpha;0,l}^{(qm)}|^2\right)^*$$

where $(\ )^*$ denotes upper semicontinuous regularization. By Lemma 4.2,

$$\left\{\left(\sup_{q\geqslant p}\frac{1}{q}\log\sum_{l=1}^{N}|\widetilde{f}_{\alpha;0,l}^{(qm)}|^2\right)^*\right\}_{p\in\mathbf{N}}$$

is a decreasing sequence of plurisubharmonic functions on W'_α which are bounded above by C'_0 on W_α, and hence $\widetilde{f}_\alpha^{(\infty)}$ is also plurisubharmonic and bounded from above by C'_0 on W_α.

Let $g_{\alpha\beta}$ and $a_{\alpha\beta} \in \mathscr{O}_X^*(W_\alpha \cap W_\beta)$, $\alpha, \beta \in I$ be the transition functions of $m(K_X + D) + L^{(m)}$ and mA respectively. By the definition of $\{f_{\alpha;0,l}^{(qm)}\}$, we have

$$f_{\alpha;0,l}^{(qm)} = (g_{\alpha\beta})^q a_{\alpha\beta} f_{\beta;0,l}^{(qm)}$$

and hence

$$\frac{1}{q}\log\sum_{l=1}^{N}|\widetilde{f}_{\alpha;0,l}^{(qm)}|^2 = \log|g_{\alpha\beta}|^2 + \frac{1}{q}\log|a_{\alpha\beta}|^2 + \frac{1}{q}\log\sum_{l=1}^{N}|\widetilde{f}_{\beta;0,l}^{(qm)}|^2.$$

Taking $\lim\limits_{p\to\infty}(\sup\limits_{q\geqslant p}\rule{1em}{0.4pt})^*$ to both sides and exponentiating them, we get rid of the term involving $a_{\alpha\beta}$ and obtain

$$e^{-\widetilde{f}_\beta^{(\infty)}} = |g_{\alpha\beta}|^2 e^{-\widetilde{f}_\alpha^{(\infty)}}.$$

This shows that the set of local data $\{e^{-\widetilde{f}_\alpha^{(\infty)}} : \alpha \in I\}$ defines a semipositive metric h_∞ on $m(K_X + D) + L^{(m)}$.

It remains to show that $\left|\sigma \wedge ds_D^{\otimes m}\right|_{h_\infty} \leqslant 1$. By Lemma 4.1,

$$\widetilde{\sigma}_{0,l}^{(qm)}|_D = \sigma^{\otimes q} \otimes t_l^{(0)} \wedge ds_D^{\otimes(qm)}$$

for $l = 1, \ldots, N$. Suppose that $\sigma \wedge ds_D^{\otimes m}$ and $t_l^{(0)}$ are represented by functions ψ_α and $\tau_{\alpha;l}^{(0)}$ on $W_\alpha \cap D$ respectively. Then we have $\widetilde{f}_{\alpha;0,l}^{(qm)} = \psi_\alpha^q \tau_{\alpha;l}^{(0)}$ for each l, and hence

$$\frac{1}{q}\log\sum_{l=1}^{N}|\widetilde{f}_{\alpha;0,l}^{(qm)}|^2\Big|_{D\cap W_\alpha} = \log|\psi_\alpha|^2 + \frac{1}{q}\log\sum_{l=1}^{N}|\tau_{\alpha;l}^{(0)}|^2.$$

Since

$$\left(\sup_{q\geqslant p}\frac{1}{q}\log\sum_{l=1}^{N}|\widetilde{f}_{\alpha;0,l}^{(qm)}|^2\right)^*\Bigg|_{D\cap W_\alpha} \geqslant \sup_{q\geqslant p}\frac{1}{q}\log\sum_{l=1}^{N}|\widetilde{f}_{\alpha;0,l}^{(qm)}|^2\Big|_{D\cap W_\alpha},$$

we obtain that

$$\widetilde{f}_\alpha^{(\infty)}|_{W_\alpha\cap D} = \lim_{p\to\infty}\left(\sup_{q\geqslant p}\frac{1}{q}\log\sum_{l=1}^{N}|\widetilde{f}_{\alpha;0,l}^{(qm)}|^2\right)^*\Bigg|_{D\cap W_\alpha} \geqslant \log|\psi_\alpha|^2.$$

This shows that $e^{-\widetilde{f}_\alpha^{(\infty)}}|\psi_\alpha|^2 \leqslant 1$ for each $\alpha \in I$ and hence completes the proof. $\square$

5. APPENDIX 1

In this appendix we will provide a proof of Theorem 3.1. Let Ω be a complex manifold and let D be a nonsingular hypersurface in Ω. Suppose that (D, h_D) and (L, h) are line bundles on Ω with singular metrics, and $s \in H^0(D, K_D + L|_D)$.

Consider the following statement:

$E\big(\Omega, (D, h_D), (L, h), s\big)$: If

(i) (L, h) is semipositive,

(ii) $h|_D$ is well defined (see 2.2 and 2.3),

(iii) there are real numbers $\mu > 0$ and $M > 0$ such that

$$\mu\sqrt{-1}\Theta_h \geqslant \sqrt{-1}\Theta_{h_D}$$

as currents on Ω and

$$\mathrm{ess.\,sup}_\Omega |s_D|_{h_D} \leqslant M,$$

and

(iv)

$$\int_D \langle s \rangle^2_{h|_D} < \infty,$$

then there is a section $\widetilde{s}_\Omega$ of $H^0(\Omega, K_\Omega + D + L)$ such that $\widetilde{s}_\Omega|_D = s \wedge ds_D$ and

$$\int_\Omega \langle \widetilde{s}_\Omega \rangle^2_{h_D \otimes h} \leqslant C \int_D \langle s \rangle^2_h$$

where $C > 0$ only depends on M and μ.

In order to simplify notations, when L and D are trivial line bundles we always write $h = e^{-\kappa}$ and $h_D = e^{-\varphi_D}$, and rewrite $E\big(\Omega, (D, h_D), (L, h), s\big)$ as $E(\Omega, \varphi_D, \kappa, s)$. For brevity, when we write $E(\Omega, \varphi_D, \kappa, s)$ we assume implicitly that D and L are trivial bundles.

Theorem 5.1. *The statement $E(Y, \varphi_D, \kappa, s)$ holds if Y is a Stein manifold and φ_D is the sum of a plurisubharmonic function and a smooth function.*

Proof of Theorem 3.1. Choose a sufficiently ample hypersurface V in X such that $D \nsubseteq V$ and D and L are trivial over $X \backslash V$ and φ_D is the sum of a plurisubharmonic function and a smooth function (cf. Remark 2.1). Then the theorem follows from Theorem 5.1 by taking $Y = X \backslash V$, and the L^2 Riemann extension theorem. $\square$

5.1. **Smoothing of singular metrics.** Let Y be a Stein manifold of dimension n. Then we can find a locally biholomorphic map $\pi : Y \to \mathbf{C}^n$ (cf. [6], p.225). In our case Y will be $X \backslash V$, the complement of an ample divisor in a projective manifold, for which such a map π can be constructed directly. The locally biholomorphic map can be used to define the operation of convolution for functions on relatively compact open subsets of Y.

We define a function $R : Y \to \mathbf{R}^+ \cup \{+\infty\}$ as follows. For $z \in \mathbf{C}^n$, denote by $B_{R'}(z)$ the ball of radius R' centered at z. For $y \in Y$, let

$$R(y) := \sup\{\, R' > 0 \mid \pi : \pi^{-1}B_{R'}\big(\pi(y)\big) \to B_{R'}\big(\pi(y)\big) \text{ is biholomorphic} \,\}.$$

R is easily seen to be lower semicontinuous. Let

$$R_\Omega := \inf_{y \in \Omega} R(y) > 0$$

for every relatively compact open subset $\Omega \Subset Y$. Note that $R_\Omega \geqslant R_{\Omega'}$ for $\Omega \Subset \Omega' \Subset Y$. If $f : Y \to \mathbf{R} \cup \{-\infty\}$ is a function and $\{\rho_\varepsilon\}$ is a family of smoothing kernels associated to a symmetric mollifier ρ on $\mathbf{C}^n$, then we can define the convolution f_ε as follows. For any $y \in Y$ we have a coordinate chart

$$\pi_y := \pi|_{U_y} : U_y := \pi^{-1}B_{R(y)}\big(\pi(y)\big) \to B_{R(y)}\big(\pi(y)\big).$$

Then

$$f_\varepsilon(y) := \big((f \circ \pi_y^{-1}) * \rho_\varepsilon\big)\big(\pi(y)\big)$$

for every y with $R(y) > \varepsilon$. Note that for any $x, y \in Y$, $f \circ \pi_x^{-1}|_{\pi(U_x \cap U_y)} = f \circ \pi_y^{-1}|_{\pi(U_x \cap U_y)}$ and hence $f_\varepsilon|_{U_y} = \big((f \circ \pi_y^{-1}) * \rho_\varepsilon\big) \circ \pi|_{U_y}$. Therefore, if f is plurisubharmonic, the convolution f_ε is also plurisubharmonic on a relatively compact open subset Ω for all $\varepsilon < R_\Omega$.

Lemma 5.1. *Suppose D_0 is a nonsingular hypersurface in Y. If D_0 and L are trivial bundles and φ_{D_0} and κ_0 are smooth on Y, then $E(\Omega, \varphi_D, \kappa, s)$ holds for every relatively compact pseudoconvex domain Ω with smooth boundary in Y and*

$$s \in \operatorname{image}\big(H^0(D_0, K_{D_0} + L|_{D_0}) \longrightarrow H^0(D, K_D + L|_D)\big),$$

where $D = D_0 \cap \Omega$, $\varphi_D = \varphi_{D_0}|_\Omega$, and $\kappa = \kappa_0|_\Omega$.

Now we deduce Theorem 5.1 from Lemma 5.1, whose proof will be given in next subsection.

Proof of Theorem 5.1. Suppose $\varphi_D = \varphi' + \varphi''$ where φ' is plurisubharmonic and φ'' is smooth, and suppose s is a section of $K_D + L|_D$ over D with

$$\int_D \langle s \rangle^2_{h|_D} < \infty.$$

Let $\varphi'_\varepsilon = \varphi' * \rho_\varepsilon$ and $\varphi''_\varepsilon = \varphi'' * \rho_\varepsilon$ on the subdomain of Ω where they can be defined. We choose a sequence of pseudoconvex domains $\Omega_1 \Subset \cdots \Subset \Omega_\nu \Subset \Omega_{\nu+1} \Subset \cdots$ with smooth boundary exhausting Y and a decreasing sequence $\{\varepsilon_\nu\}$ converging to zero such that the following conditions hold:

(1) $R_{\Omega_\nu} > \varepsilon_\nu$ and $\kappa_{\varepsilon_\nu} = \kappa * \rho_{\varepsilon_\nu}$ is a smooth plurisubharmonic function on Ω_ν.
(2) For each $N \in \mathbf{N}$, the sequences $\{\kappa_{\varepsilon_\nu}\}_{\nu \geqslant N}$ and $\{\varphi'_{\varepsilon_\nu}\}_{\nu \geqslant N}$ decrease to κ and φ' on Ω_N, respectively.
(3) For each ν, we have $|s_D|^2 e^{-\varphi''_{\varepsilon_\nu}} \leqslant 2|s_D|^2 e^{-\varphi''}$ on Ω_ν. (Here $|s_D|^2$ is taken by viewing s_D as a function via the global trivialization of D. Note that on each relatively compact set $e^{-\varphi''_\varepsilon}$ converges to $e^{-\varphi''}$ as $\varepsilon \to 0$. Therefore for each ν we only need to choose ε_ν so small that $|e^{-\varphi''_{\varepsilon_\nu}} - e^{-\varphi''}| \leqslant \inf_{\Omega_\nu} e^{-\varphi''}$ on Ω_ν.)

Therefore

$$\sup_{\Omega_\nu} |s_D|_{\varphi_{\varepsilon_\nu}} \leqslant \sqrt{2}\,\mathrm{ess.}\sup_{\Omega} |s_D|_{\varphi_D} \leqslant \sqrt{2}M$$

for each ν. Clearly, κ_{ε_ν} is not identically $-\infty$ on $D \cap \Omega_\nu$.

The curvature condition

$$\mu\sqrt{-1}\Theta_\kappa \geqslant \sqrt{-1}\Theta_{\varphi_D}$$

implies that there is a plurisubharmonic function ψ such that $\mu\kappa - \varphi_D = \psi$ a.e. on Y. Then $\mu\kappa_{\varepsilon_\nu} - \varphi_{\varepsilon_\nu} = \psi * \rho_{\varepsilon_\nu}$ a.e. on Ω_ν. Since $\psi * \rho_{\varepsilon_\nu}$ is plurisubharmonic, we get

$$\mu\sqrt{-1}\Theta_{\kappa_{\varepsilon_\nu}} \geqslant \sqrt{-1}\Theta_{\varphi_{\varepsilon_\nu}}$$

on Ω_ν. Having assumed the validity of Lemma 5.1, we can obtain such an extension $\widetilde{s}_{\Omega_\nu}$. Since $\kappa_{\varepsilon_\nu} \geqslant \kappa$, we obtain

$$\int_{\Omega_N} \langle \widetilde{s}_{\Omega_\nu} \rangle^2_{\varphi_{\varepsilon_\nu}+\kappa_{\varepsilon_\nu}} \leqslant C \int_{D\cap\Omega_N} \langle s \rangle^2_{\kappa_{\varepsilon_\nu}} \leqslant C \int_D \langle s \rangle^2_\kappa \tag{5.1}$$

for all $\nu \geqslant N$. (Here we abuse the notation by using weight functions to stand for their associated metrics.) Notice that the RHS is independent of n (C only depends on M and μ). By (iii) in $E(Y, \varphi_D, \kappa, s)$, for each $N \in \mathbf{N}$, the weight function $\varphi + \kappa$ is bounded from above on Ω_{N+1} by a number $M_N > 0$. By the definition of convolution $\varphi_{\varepsilon_\nu} + \kappa_{\varepsilon_\nu}$ are bounded from above by the same number M_N on Ω_N for sufficiently large ν. By diagonal method we can select a subsequence $\{\widetilde{s}_{\Omega_{\nu_k}}\}_{k\in\mathbf{N}}$ such that $\{\widetilde{s}_{\Omega_{\nu_k}}\}_{k\geqslant N}$ converges uniformly on Ω_N for each $N \in \mathbf{N}$. This way we obtain a section $\widetilde{s}_Y \in H^0(Y, K_Y + D + L)$ by setting $\widetilde{s}_Y|_{\Omega_{\nu_N}} = \lim_{k\to\infty} \widetilde{s}_{\Omega_{\nu_k}}$. We let χ_{Ω_N} be the characteristic function of Ω_N on Y. (5.1) can be rephrased as

$$\int_Y \chi_{\Omega_N} \langle \widetilde{s}_{\Omega_\nu} \rangle^2_{\varphi_{\varepsilon_\nu}+\kappa_{\varepsilon_\nu}} \leqslant C \int_D \langle s \rangle^2_\kappa.$$

Applying Fatou's lemma, we obtain the desired inequality

$$\int_Y \langle \widetilde{s}_Y \rangle^2_{\varphi+\kappa} \leqslant C \int_D \langle s \rangle^2_\kappa.$$

□

The rest of this appendix is devoted to proving Lemma 5.1.

5.2. **Proof of Lemma 5.1.** From now on, we assume that Y, Ω and s are as in Lemma 5.1. Let ρ be a defining function of Ω. We follow almost the same argument as Siu's in [19].

Definition 5.1. Let $(z^1, \dots, z^n)$ be local coordinates on some open set U and let e_U be a local holomorphic frame of L. We put $e^{-\psi} = h(e_U, e_U)$.

(1) For u, v being L-valued (p,q)-forms with measurable coefficients, we set

$$\langle u, v \rangle_h := \langle u, v \rangle_{g_\omega \otimes h} dV_\omega$$

and $|u|^2_h := \langle u, u \rangle_h$ where g is a hermitian metric on Ω with ω being its associated $(1,1)$-form. We will sometimes write $\langle u, v \rangle_\psi = \langle u, v \rangle_h$ by abusing the notation. Note that when $(p,q) = (n,0)$ we have $|u|^2_h = \langle u \rangle^2_h$ as in Definition 2.1.

(2) Given an L-valued $(n,1)$-form u. Locally we have $u = \sum_\beta u_{\overline{\beta}} e_U \otimes dz \wedge d\overline{z}^\beta$ where $dz := dz^1 \wedge \cdots \wedge dz^n$. We define an $(n,0)$-form

$$\iota^\alpha u := \sum_\beta g^{\alpha\overline{\beta}} u_{\overline{\beta}} e_U \otimes dz.$$

For a continuous $(1,1)$-form Ξ which has a local expression $\sqrt{-1}\xi_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$. We set $\Xi[u]_h := \sum_{\alpha,\beta} \xi_{\alpha\overline{\beta}} \left\langle \iota^\alpha u, \iota^\beta u \right\rangle_\psi$.

We also need the following standard result from functional analysis:

Lemma 5.2. *Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ and $S : \mathcal{H}_2 \to \mathcal{H}_3$ be closed, densely defined operators between Hilbert spaces with $ST = 0$, and let $C > 0$ be a constant. Given $g \in \mathcal{H}_2$ with $Sg = 0$. Then there exists $v \in \mathcal{H}_1$ such that $Tv = g$ and $\|v\| \leqslant C$ if and only if*

$$|(u,g)|^2 \leqslant C^2(\|T^*u\|^2 + \|Su\|^2) \tag{5.2}$$

for all $u \in \operatorname{Dom} S \cap \operatorname{Dom} T^*$.

Let φ, η and γ be smooth functions with $\eta, \gamma > 0$. Set $\eta e^{-\psi} = e^{-\varphi}$. We recall the twisted Bochner–Kodaira formula (see [19], Proposition 3.4)

$$\begin{aligned} \int_\Omega |\overline{\partial}^*_\psi u|^2_\varphi + \int_\Omega |\overline{\partial} u|^2_\varphi &= \int_{\partial\Omega} \sqrt{-1}\partial\overline{\partial}\rho_\Omega[u]_\varphi + \int_\Omega |\nabla^{0,1} u|^2_\varphi \\ &+ \int_\Omega \left(\eta\sqrt{-1}\partial\overline{\partial}\psi - \sqrt{-1}\partial\overline{\partial}\eta\right)[u]_\psi + 2\mathrm{Re}\int_\Omega \left\langle \iota^{\partial\eta} u, \overline{\partial}^*_\psi u \right\rangle_\psi \end{aligned} \tag{5.3}$$

for each L-valued smooth $(n,1)$-form u in $\operatorname{Dom} \overline{\partial}^*_\psi \cap \operatorname{Dom} \overline{\partial}$.

Remark 5.2. For $\mathcal{E}^{n,1}_c(\overline{\Omega}, L)$ being the space of L-valued smooth $(n,1)$-forms with compact supports, $\mathcal{E}^{n,1}_c(\overline{\Omega}, L) \cap \operatorname{Dom} \overline{\partial}^*_\psi \cap \operatorname{Dom} \overline{\partial}$ is dense in $\operatorname{Dom} \overline{\partial}^*_\psi \cap \operatorname{Dom} \overline{\partial}$ with respect to the graph norm. Therefore, to get a priori estimate from Lemma 5.2 we only need to consider smooth u with compact supports.

Since Ω is pseudoconvex, the Levi form of ρ_Ω is semipositive at each point of $\partial\Omega$. Adding $\int_\Omega \gamma |\overline{\partial}^*_\psi u|^2_\psi$ to both side of (5.3) and using $\eta e^{-\psi} = e^{-\varphi}$, the twisted Bochner–Kodaira formula becomes

$$\begin{aligned} &\int_\Omega (\eta+\gamma) |\overline{\partial}^*_\psi u|^2_\psi + \int_\Omega \eta |\overline{\partial} u|^2_\psi \\ &\geqslant \int_\Omega \left(\eta\sqrt{-1}\partial\overline{\partial}\psi - \sqrt{-1}\partial\overline{\partial}\eta\right)[u]_\psi + 2\mathrm{Re}\int_\Omega \left\langle \iota^{\partial\eta} u, \overline{\partial}^*_\psi u \right\rangle_\psi + \int_\Omega \gamma |\overline{\partial}^*_\psi u|^2_\psi. \end{aligned} \tag{5.4}$$

We set $r(x) := |s_D(x)|_{h_D}$ for $x \in \Omega$. We first assume that $\frac{1}{\mu} \geqslant 2M^2$ and let c be a positive constant to be specified later. We set $N_0 := \max\{1, \sqrt{e}M^{2c}\}$. Choose any positive number $A > N_0$. Let

$$\varepsilon_0 = \sqrt{\left(\frac{A}{\sqrt{e}}\right)^{1/c} - M^2}.$$

For each positive $\varepsilon \leqslant \varepsilon_0$, we let

$$\eta = \log \frac{A}{(r^2+\varepsilon^2)^c}$$

and

$$\gamma = \frac{2c^2}{r^2+\varepsilon^2}.$$

Then $\eta \geqslant 1/2$ on Ω. Applying the Cauchy–Schwarz inequality and $\partial\eta = -\frac{2cr}{r^2+\varepsilon^2}\partial r$, we obtain

$$\begin{aligned}\left|2\mathrm{Re}\int_\Omega \left\langle \iota^{\partial\eta}u, \overline{\partial}^*_\psi u\right\rangle_\psi\right| &\leqslant 2\int_\Omega |\iota^{\partial\eta}u|_\psi|\overline{\partial}^*_\psi u|_\psi\\ &= 2\int_\Omega \frac{2cr}{r^2+\varepsilon^2}|\iota^{\partial r}u|_\psi|\overline{\partial}^*_\psi u|_\psi\\ &= \int_\Omega \frac{2r^2}{r^2+\varepsilon^2}|\iota^{\partial r}u|^2_\psi + \int_\Omega \frac{2c^2}{r^2+\varepsilon^2}|\overline{\partial}^*_\psi u|^2_\psi\\ &= \int_\Omega \frac{2r^2}{r^2+\varepsilon^2}|\iota^{\partial r}u|^2_\psi + \int_\Omega \gamma|\overline{\partial}^*_\psi u|^2_\psi.\end{aligned}$$

From (5.4) it follows that

$$\begin{aligned}&\int_\Omega(\eta+\gamma)|\overline{\partial}^*_\psi u|^2_\psi + \int_\Omega \eta|\overline{\partial}u|^2_\psi\\ &\qquad\geqslant \int_\Omega\left(\eta\sqrt{-1}\partial\overline{\partial}\psi - \sqrt{-1}\partial\overline{\partial}\eta\right)[u]_\psi - \int_\Omega\frac{2r^2}{r^2+\varepsilon^2}|\iota^{\partial r}u|^2_\psi\end{aligned}\tag{5.5}$$

Now we compute $-\partial\overline{\partial}\eta$. Since $r^2\partial\overline{\partial}\log r^2 = 2r\partial\overline{\partial}r - 2\partial r\wedge\overline{\partial}r$, it follows that

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}r^2 &= 2\sqrt{-1}\partial r\wedge\overline{\partial}r + 2r\sqrt{-1}\partial\overline{\partial}r\\ &= r^2\sqrt{-1}\partial\overline{\partial}\log r^2 + 4\sqrt{-1}\partial r\wedge\overline{\partial}r.\end{aligned}$$

By the Poincaré–Lelong formula,

$$\sqrt{-1}\partial\overline{\partial}\log r^2 = 2\pi[D] - \sqrt{-1}\partial\overline{\partial}\varphi_D,$$

where $[D]$ is the current of integration over D. Hence

$$\sqrt{-1}\partial\overline{\partial}r^2 = 4\sqrt{-1}\partial r\wedge\overline{\partial}r - r^2\sqrt{-1}\partial\overline{\partial}\varphi_D.\tag{5.6}$$

The term involving the current of integration vanishes since $r^2 \equiv 0$ on D.

We let $\eta_0 = -\log(r^2+\varepsilon^2)$. From $\partial\overline{\partial}r^2 = \partial\overline{\partial}(e^{-\eta_0})$ it follows that

$$\begin{aligned}\partial\overline{\partial}r^2 &= e^{-\eta_0}\left(\partial\eta_0\wedge\overline{\partial}\eta_0 - \partial\overline{\partial}\eta_0\right)\\ &= \frac{4r^2}{r^2+\varepsilon^2}\partial r\wedge\overline{\partial}r - (r^2+\varepsilon^2)\partial\overline{\partial}\eta_0.\end{aligned}\tag{5.7}$$

Using (5.6), (5.7) and $\partial\overline{\partial}\eta = c\partial\overline{\partial}\eta_0$, we get

$$-\sqrt{-1}\partial\overline{\partial}\eta = -\frac{cr^2}{r^2+\varepsilon^2}\sqrt{-1}\partial\overline{\partial}\varphi_D + \frac{4c\varepsilon^2}{(r^2+\varepsilon^2)^2}\sqrt{-1}\partial r\wedge\overline{\partial}r.\tag{5.8}$$

Choose $\psi = \kappa + \frac{r^2}{2\mu M^2}$. Note that $\sqrt{-1}\partial\overline{\partial}\psi \geqslant 0$. Using (5.6) and (5.8) we get

$$\begin{aligned}&\eta\sqrt{-1}\partial\overline{\partial}\psi - \sqrt{-1}\partial\overline{\partial}\eta = \eta\sqrt{-1}\partial\overline{\partial}\kappa\\ &\quad -\eta\left(\frac{r^2}{2\mu M^2} + \frac{cr^2}{\eta(r^2+\varepsilon^2)}\right)\sqrt{-1}\partial\overline{\partial}\varphi_D + \left(\frac{4\eta}{2\mu M^2} + \frac{4c\varepsilon^2}{(r^2+\varepsilon^2)^2}\right)\sqrt{-1}\partial r\wedge\overline{\partial}r.\end{aligned}\tag{5.9}$$

Since $\eta \geqslant 1/2$ we get

$$\frac{r^2}{2\mu M^2} + \frac{cr^2}{\eta\,(r^2+\varepsilon^2)} \leqslant \frac{1}{2\mu} + 2c. \tag{5.10}$$

We now choose c so that $c \leqslant \frac{1}{4\mu}$. It follows that

$$\sqrt{-1}\partial\overline{\partial}\kappa - \left(\frac{r^2}{2\mu M^2} + \frac{cr^2}{\eta\,(r^2+\varepsilon^2)}\right)\sqrt{-1}\partial\overline{\partial}\varphi_D \geqslant 0 \tag{5.11}$$

where the inequality is from (5.10) and the curvature hypothesis.

From (5.5), (5.9), (5.11) and $\frac{4\eta}{2\mu M^2} \geqslant 2 \geqslant \frac{2r^2}{r^2+\varepsilon^2}$ we obtain

$$\int_\Omega (\eta+\gamma)\,|\overline{\partial}^*_\psi u|^2_\psi + \int_\Omega \eta|\overline{\partial}u|^2_\psi \geqslant \int_\Omega \frac{4c\varepsilon^2}{(r^2+\varepsilon^2)^2}|\iota^{\partial r}u|^2_\psi. \tag{5.12}$$

We now consider the modified $\overline{\partial}$ operators T and S defined by

$$Tu = \overline{\partial}\left(\sqrt{\eta+\gamma}u\right) \qquad \text{and} \qquad Su = \sqrt{\eta}(\overline{\partial}u),$$

respectively. They are densely defined and $S \circ T = 0$, and we can rewrite (5.12) to obtain the following lemma. (See Remark 5.2.)

Lemma 5.3. *For each L-valued $(n,1)$-form u in* $\operatorname{Dom} S \cap \operatorname{Dom} T^*$ *we have*

$$\|T^*u\|^2_{\Omega,\psi} + \|Su\|^2_{\Omega,\psi} \geqslant \int_\Omega \frac{4c\varepsilon^2}{(r^2+\varepsilon^2)^2}|\iota^{\partial r}u|^2_\psi. \tag{5.13}$$

Here $\|\cdot\|_{\Omega,\psi}$ means the L^2 norm for Ω with respect to the weight function $e^{-\psi}$.

Since Y is Stein, there exists a $(D+L)$-valued n-form $\widetilde{s}_0$ on Y such that $\widetilde{s}_0|_D = s \wedge ds_D$. Choose any number $0 < \delta < 1$. Let $\varrho \in C^\infty([0,+\infty))$ be a cut-off function with $0 \leqslant \varrho(x) \leqslant 1$ so that ϱ is identically 1 on $[0, \frac{\delta}{2}]$ and

$$\operatorname{supp}\varrho \subseteq [0,1] \quad \text{and} \quad \sup|\varrho'| \leqslant 1+\delta. \tag{5.14}$$

Let $\varrho_\varepsilon := \varrho\left(\frac{r^2}{\varepsilon^2}\right)$ and let

$$\alpha_\varepsilon := \frac{2r}{\varepsilon^2}\,\varrho'\left(\frac{r^2}{\varepsilon^2}\right)\overline{\partial}r \wedge \left(s_D^{-1}\otimes\widetilde{s}_0\right).$$

Note that α_ε is smooth because the singularity of $\overline{\partial}r \wedge \left(s_D^{-1}\otimes\widetilde{s}_0\right)$ lies in the zero locus D of s_D and $\varrho'\left(\frac{r^2}{\varepsilon^2}\right)$ equals zero in the tubular neighborhood $r^2 < \frac{\delta}{2}\varepsilon^2$. Then

we have

$$
\begin{aligned}
|(u,\alpha_\varepsilon)_{\Omega,\psi}|^2 &= \left(\int_\Omega \left|\left\langle u, \frac{2r}{\varepsilon^2}\,\varrho'\left(\frac{r^2}{\varepsilon^2}\right)\overline{\partial}r\wedge\left(s_D^{-1}\otimes\widetilde{s}_0\right)\right\rangle_\psi\right|\right)^2 \\
&= \left(\int_\Omega \left|\left\langle \iota^{\partial r}u, \frac{2r}{\varepsilon^2}\,\varrho'\left(\frac{r^2}{\varepsilon^2}\right)s_D^{-1}\otimes\widetilde{s}_0\right\rangle_\psi\right|\right)^2 \\
&\leqslant \left(\int_\Omega 2|\iota^{\partial r}u|_\psi \left|\frac{r}{\varepsilon^2}\,\varrho'\left(\frac{r^2}{\varepsilon^2}\right)s_D^{-1}\otimes\widetilde{s}_0\right|_\psi\right)^2 \\
&\leqslant \left(\int_\Omega \left|\frac{r}{\varepsilon^2}\,\varrho'\left(\frac{r^2}{\varepsilon^2}\right)s_D^{-1}\otimes\widetilde{s}_0\right|^2_\psi \frac{(r^2+\varepsilon^2)^2}{c\varepsilon^2}\right) \\
&\quad\times\left(\int_\Omega \frac{4c\varepsilon^2}{(r^2+\varepsilon^2)^2}|\iota^{\partial r}u|^2_\psi\right) \\
&\leqslant C_\varepsilon\left(\|T^*u\|^2_{\Omega,\psi}+\|Su\|^2_{\Omega,\psi}\right),
\end{aligned}
$$

where the last inequality is from Lemma 5.3, and we have used the notation

$$
C_\varepsilon := \int_\Omega \left|\frac{r}{\varepsilon^2}\,\varrho'\left(\frac{r^2}{\varepsilon^2}\right)s_D^{-1}\otimes\widetilde{s}_0\right|^2_\psi \frac{(r^2+\varepsilon^2)^2}{c\varepsilon^2}.
$$

By Lemma (5.2), we can solve the equation $T\beta_\varepsilon = \bar{\partial}\left(\sqrt{\eta+\gamma}\beta_\varepsilon\right) = \alpha_\varepsilon$ such that

$$
\int_\Omega |\beta_\varepsilon|^2_\psi \leqslant C_\varepsilon. \tag{5.15}
$$

5.3. **Estimate the constant C_ε.** Now we estimate the constant C_ε. Take $y\in Y$ an arbitrary point and $(z^j = x^j + iy^j)$ local coordinates on a open set U_y centered at y, and let e_L (respectively, e_D) be local frames of L (respectively, D) such that the following conditions holds:

(1) $s_D = z^n\otimes e_D$ on U_y;
(2) If $\zeta = \xi + i\tau := z^n e^{-\frac{\varphi_D}{2}}$, then $(x^1, y^1, \cdots, \xi, \tau)$ forms a coordinate system;
(3) $U_y = P_{n-1}\times\{r<\varepsilon\}$ where P_{n-1} is a $(n-1)$-dimensional polydisc;
(4) We have

$$
\widetilde{s}_0 = \widetilde{\sigma}_{U_y} e_D\otimes e_L\otimes dz^1\wedge\cdots\wedge dz^n \text{ and } s = \sigma_{U_y} e_L\otimes dz^1\wedge\cdots\wedge dz^{n-1}
$$

on U_y. (Note that $\widetilde{s}_0$ is defined not only on Ω but on Y.)

Since $\widetilde{s}_0|_D = s\wedge ds_D$, we get

$$
\widetilde{\sigma}_{U_y}(z^1,\cdots,z^{n-1},0) = \sigma_{U_y}(z^1,\cdots,z^{n-1})
$$

on U_y. Choose a partition of unity $\{\rho_j\}$ subordinate to a finite subcover $\{U_j\}\subset\{U_y\}_{y\in\overline{\Omega}}$ of $\overline{\Omega}$. Then

$$
\begin{aligned}
C_\varepsilon &= \int_\Omega \frac{(r^2+\varepsilon^2)^2}{c\varepsilon^6} r^2\left|\varrho'\left(\frac{r^2}{\varepsilon^2}\right)\right|^2 |s_D^{-1}\otimes\widetilde{s}_0|^2_\psi \\
&\leqslant \frac{(1+\delta)^2}{c}\int_{\Omega\cap\{\sqrt{\frac{\delta}{2}}\varepsilon\leqslant r\leqslant\varepsilon\}} \frac{(r^2+\varepsilon^2)^2}{\varepsilon^6} r^2 |s_D^{-1}\otimes\widetilde{s}_0|^2_\psi.
\end{aligned}
$$

For each j we let $V_j := U_j \cap \Omega \cap \{\sqrt{\frac{\delta}{2}}\varepsilon \leqslant r \leqslant \varepsilon\}$ and

$$\begin{aligned} I_j &= \int_{V_j} \rho_j \frac{(r^2+\varepsilon^2)^2}{\varepsilon^6} r^2 |s_D^{-1} \otimes \widetilde{s}_0|^2_\psi \\ &= \int_{V_j} \rho_j \frac{(r^2+\varepsilon^2)^2}{\varepsilon^6} \big|\widetilde{\sigma}_{U_j}\big|^2 e^{-\kappa - \frac{r^2}{2\mu M^2}} e^{-\varphi_D} dx^1 \wedge dy^1 \wedge \cdots \wedge dy^n. \end{aligned}$$

Therefore

$$C_\varepsilon \leqslant \frac{(1+\delta)^2}{c} \sum\nolimits_j I_j.$$

A direct computation yields

$$\begin{aligned} & e^{-\varphi_D} dx^1 \wedge dy^1 \wedge \cdots \wedge dy^n \\ &= (1+O(r))\, dx^1 \wedge dy^1 \wedge \cdots \wedge dy^{n-1} \wedge d\xi \wedge d\tau. \end{aligned}$$

Let $f := \rho_j \big|\widetilde{\sigma}_{U_j}\big|^2 e^{-\kappa}$. Then

$$\begin{aligned} I_j &\leqslant \int_{V_j} \rho_j \big|\widetilde{\sigma}_{U_j}\big|^2 e^{-\kappa} \frac{(r^2+\varepsilon^2)^2}{\varepsilon^6} (1+O(r)) r\, dx^1 dy^1 \cdots dy^{n-1} dr\, d\theta \\ &= \mathrm{I}^{(j)} + \mathrm{II}^{(j)} + \mathrm{III}^{(j)}, \end{aligned}$$

where

$$\mathrm{I}^{(j)} = \int_{P_{n-1} \cap D} f(z^1, \cdots, z^{n-1}, 0)\, dx^1 dy^1 \cdots dy^{n-1} \left(2\pi \int_0^\varepsilon \frac{(r^2+\varepsilon^2)^2}{\varepsilon^6} r\, dr \right),$$

$$\mathrm{II}^{(j)} = \int_{V_j} (f(z^1, \cdots, z^n) - f(z^1, \cdots, 0)) \frac{(r^2+\varepsilon^2)^2}{\varepsilon^6} r\, dx^1 dy^1 \cdots dr\, d\theta,$$

$$\mathrm{III}^{(j)} = \int_{V_j} f(z^1, \cdots, z^n) \frac{(r^2+\varepsilon^2)^2}{\varepsilon^6} O(r) r\, dx^1 dy^1 \cdots dr\, d\theta.$$

Note that the term $f(z^1, \cdots, z^n) - f(z^1, \cdots, 0)$ in $\mathrm{II}^{(j)}$ produces a factor r. Thus $\mathrm{II}^{(j)}$ and $\mathrm{III}^{(j)}$ converge to zero as ε tend 0^+. Then

$$\begin{aligned} \limsup_{\varepsilon \to 0^+} C_\varepsilon &\leqslant \frac{(1+\delta)^2}{c} \sum\nolimits_j \mathrm{I}^{(j)} \\ &= \frac{2\pi(1+\delta)^2}{c} \Big(\int_{\Omega \cap D} \langle s \rangle_h^2 \Big) \limsup_{\varepsilon \to 0^+} \int_0^\varepsilon \frac{(r^2+\varepsilon^2)^2}{\varepsilon^6} r\, dr \\ &= \frac{7\pi}{3c} (1+\delta)^2 \int_{\Omega \cap D} \langle s \rangle_h^2. \end{aligned} \tag{5.16}$$

5.4. **Extension and the L^2 norm bound.** Now we set

$$\widetilde{S}_\varepsilon := \varrho_\varepsilon \widetilde{s}_0 - \sqrt{\eta + \gamma}\, (s_D \otimes \beta_\varepsilon).$$

Then $\widetilde{S}_\varepsilon$ is a holomorphic section by construction and

$$\int_\Omega |\varrho_\varepsilon \widetilde{s}_0|^2_{h_D \otimes h} \to 0 \text{ as } \varepsilon \to 0^+,$$

because $\widetilde{s}_0$ is smooth in the relatively compact set Ω and the support of $\varrho_\varepsilon \widetilde{s}_0$ approaches a set of measure zero in Ω as $\varepsilon \to 0^+$.

The supremum norm of $r\sqrt{\eta+\gamma}$ on $\Omega \subseteq \{r \leqslant M\}$ is no more than the square root of

$$\sup_{0<x\leqslant M} x^2 \left(\log A + c \log \frac{1}{x^2+\varepsilon^2} + \frac{2c^2}{x^2+\varepsilon^2}\right) \leqslant M^2 \log A + \frac{c}{e} + 2c^2,$$

because the maximum of $y \log \frac{1}{y}$ on $(0, +\infty)$ occurs at $y = \frac{1}{e}$.

Take $A \to N_0^+$, $\delta \to 0^+$. By using (5.16) and

$$\int_\Omega |\beta_\varepsilon|_h^2 \leqslant e^{\frac{1}{2\mu}} C_\varepsilon$$

from (5.15) and $r \leqslant M$, we get

$$\limsup_{\varepsilon \to 0^+} \int_\Omega \langle \widetilde{S}_\varepsilon \rangle_{h_D \otimes h}^2 \leqslant C_0 \int_\Omega \langle s \rangle_h^2$$

where $C_0 = \frac{7\pi}{3} e^{\frac{1}{2\mu}} \sqrt{\left(\frac{M}{c}\right)^2 \log N_0 + \frac{1}{ce} + 2}$. Then the limit $\widetilde{s}_\Omega$ (up to subsequences) is an $D + L$-valued holomorphic n-form on Ω whose restriction to D is $s \wedge ds_D$ with the following estimate

$$\int_\Omega \langle \widetilde{s}_\Omega \rangle_{h_D \otimes h}^2 \leqslant C_0 \int_D \langle s \rangle_h^2.$$

If $\frac{1}{\mu} < 2M^2$, we replace the metric h_D by the metric $h'_D := \frac{1}{2\mu M^2} h_D$. Then $\sup_\Omega |T|_{h'_D} = \frac{1}{\sqrt{2\mu}}$. This finishes the proof of Theorem 5.1.

Remark 5.3. In the statement of Theorem 5.1 the requirement that D and L being trivial bundles is used only for smoothing the metrics on them. Therefore the same argument shows that $E\big(\Omega, (D, h_D), (L, h), s\big)$ holds if Ω is Stein, (D, h_D) and (L, h) are smoothly metrized, and $s \in H^0(D, K_D + L|_D)$.

6. Appendix 2

The following lemma about generalized multiplication maps is used in 3.2 to select the auxiliary ample divisor to fulfill (A_3). For the convenience of the readers we give a proof in this appendix. Some of its special cases are well known in [9], [16]. The proof presented below is a modification of their arguments.

Lemma 6.1. *Let D and E be ample Cartier divisors on a scheme X. For any coherent sheaves $\mathscr{F}_1$ and $\mathscr{F}_2$ on X, there is a positive integer $m_0 = m_0(D, E, \mathscr{F}_1, \mathscr{F}_2)$ such that*

$$H^0\big(X, \mathscr{F}_1 \otimes \mathscr{O}_X(aD)\big) \otimes H^0\big(X, \mathscr{F}_2 \otimes \mathscr{O}_X(bE)\big) \to H^0\big(X, \mathscr{F}_1 \otimes \mathscr{F}_2 \otimes \mathscr{O}_X(aD + bE)\big)$$

is surjective for all $a, b \geqslant m_0$.

Proof. First we assume that $\mathscr{F}_1$ and $\mathscr{F}_2$ are locally free. Consider on $X \times X$ the exact sequence

$$0 \longrightarrow \mathscr{I}_\Delta \longrightarrow \mathscr{O}_{X \times X} \longrightarrow \Delta_* \mathscr{O}_X \longrightarrow 0 \tag{6.1}$$

where Δ is the diagonal morphism. Let p_1 and p_2 be the two projections and

$$aD \boxplus bE = p_1^*(aD) \otimes p_2^*(bE)$$

and

$$\mathscr{G} = p_1^* \mathscr{F}_1 \otimes p_2^* \mathscr{F}_2.$$

By tensoring (6.1) with $\mathscr{G}$, we get

$$0 \longrightarrow \mathscr{G} \otimes \mathscr{I}_\Delta \longrightarrow \mathscr{G} \longrightarrow \mathscr{G} \otimes \Delta_* \mathscr{O}_X \longrightarrow 0.$$

Twisting by $\mathscr{O}_{X \times X}(aD \boxplus bE)$ and taking cohomology, we obtain an exact sequence

$$H^0\big(X \times X, \mathscr{G}(a,b)\big) \to H^0\big(X \times X, (\mathscr{G} \otimes \Delta_* \mathscr{O}_X)(a,b)\big) \to H^1\big(X \times X, (\mathscr{G} \otimes \mathscr{I}_\Delta)(a,b)\big)$$

where we use (a,b) to denote the twisting $\otimes \mathscr{O}_{X \times X}(aD \boxplus bE)$. It suffices to verify that there is a positive integer m_0 such that

$$H^1\big(X \times X, \big(\mathscr{G} \otimes \mathscr{I}_\Delta\big)(a,b)\big) = 0. \tag{6.2}$$

for $a, b \geqslant m_0$. Indeed, there is an isomorphism of cohomology groups

$$H^0\big(X \times X, \mathscr{G}(a,b)\big) \cong H^0\big(X, \mathscr{F}_1 \otimes \mathscr{O}_X(aD)\big) \otimes H^0\big(X, \mathscr{F}_2 \otimes \mathscr{O}_X(bE)\big).$$

By the projection formula,

$$\big(\mathscr{G} \otimes \Delta_* \mathscr{O}_X\big)(a,b) \cong \Delta_*\big(\Delta^* \mathscr{G}(a,b)\big).$$

By definition of the diagonal morphism we have $p_i \Delta = id_X$, hence

$$\Delta^*\big(p_1^* \mathscr{F}_1 \otimes p_2^* \mathscr{F}_2 \otimes p_1^*(aD) \otimes p_2^*(bE)\big) \cong \mathscr{F}_1 \otimes \mathscr{F}_2 \otimes \mathscr{O}_X(aD + bE).$$

Therefore the cohomology group

$$H^0\big(X \times X, (\mathscr{G} \otimes \Delta_* \mathscr{O}_X)(a,b)\big) \cong H^0\big(X \times X, \Delta_* \Delta^* \mathscr{G}(a,b)\big)$$

is isomorphic to

$$H^0\big(X, \Delta^* \mathscr{G}(a,b)\big) \cong H^0\big(X, \mathscr{F}_1 \otimes \mathscr{F}_2 \otimes \mathscr{O}_X(aD + bE)\big)$$

as desired.

Now we prove (6.2). To this end, we use the ample divisor $aD \boxplus bE$ to construct a (possibly non-terminating) resolution

$$\cdots \longrightarrow \bigoplus \mathscr{O}_{X \times X}(-p_1, -p_1) \longrightarrow \bigoplus \mathscr{O}_{X \times X}(-p_0, -p_0) \longrightarrow \mathscr{G} \otimes \mathscr{I}_\Delta \longrightarrow 0 \tag{6.3}$$

for suitable integers $0 \leqslant p_0 \leqslant p_1 \leqslant \cdots$ where again (a,b) means the twisting $\otimes \mathscr{O}_{X \times X}(aD \boxplus bE)$. Set $d = \dim X \times X$. By dimension shifting, to prove (6.2) it is enough to produce an integer m_0 such that

$$H^i\big(X \times X, \mathscr{O}_{X \times X}(a - p_{i-1}, b - p_{i-1})\big) = 0$$

whenever $a, b \geqslant m_0$ and $i = 0, 1, \ldots, d-1$. In fact, we have then

$$\begin{aligned} H^1\big(X \times X, \big(\mathscr{G} \otimes \mathscr{I}_\Delta\big)(a,b)\big) &\cong H^2\big(X \times X, \mathscr{K}_0(a,b)\big) \\ &\;\;\vdots \\ &\cong H^d\big(X \times X, \mathscr{K}_{d-2}(a,b)\big) \\ &\cong H^{d+1}\big(X \times X, \mathscr{K}_{d-1}(a,b)\big) = 0 \end{aligned}$$

where $\mathscr{K}_i$ is the kernel of the morphism

$$\bigoplus \mathscr{O}_{X\times X}(-p_i,-p_i) \to \bigoplus \mathscr{O}_{X\times X}(-p_{i-1},-p_{i-1})$$

for $i>0$ and $\mathscr{K}_0$ is the kernel of

$$\bigoplus \mathscr{O}_{X\times X}(-p_0,-p_0) \to \mathscr{G}\otimes \mathscr{I}_\Delta \to 0.$$

The last group vanishes by dimension reason. The existence of the required integer m_0 then follows from Serre's vanishing theorem.

For general coherent sheaves $\mathscr{F}_j$, we can write $\mathscr{F}_j$ as a quotient of a sheaf $\mathscr{E}_j$ which is a finite direct sum of sheaves of the form $\mathscr{O}_X(q_i)$. We consider the following exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{E}_1\otimes\mathscr{E}_2 \longrightarrow \mathscr{F}_1\otimes\mathscr{F}_2 \longrightarrow 0.$$

Choose a positive integer m_0 such that

(1) $H^1(X,\mathscr{K}\otimes\mathscr{O}_X(aD+bE))$ vanishes for $a,b\geqslant m_0$, and
(2) the multiplication map

$$H^0\big(X,\mathscr{E}_1\otimes\mathscr{O}_X(aD)\big)\otimes H^0\big(X,\mathscr{E}_2\otimes\mathscr{O}_X(bE)\big) \to H^0\big(X,\mathscr{E}_1\otimes\mathscr{E}_2\otimes\mathscr{O}_X(aD+bE)\big)$$

is surjective whenever $a,b\geqslant m_0$.

Consider the commutative diagram

$$\begin{tikzcd}
H^0\big(X,\mathscr{E}_1\otimes\mathscr{O}_X(aD)\big)\otimes H^0\big(X,\mathscr{E}_2\otimes\mathscr{O}_X(bE)\big) \arrow[r] \arrow[d] & H^0\big(X,\mathscr{E}_1\otimes\mathscr{E}_2\otimes\mathscr{O}_X(aD+bE)\big) \arrow[d] \\
H^0\big(X,\mathscr{F}_1\otimes\mathscr{O}_X(aD)\big)\otimes H^0\big(X,\mathscr{F}_2\otimes\mathscr{O}_X(bE)\big) \arrow[r] & H^0\big(X,\mathscr{F}_1\otimes\mathscr{F}_2\otimes\mathscr{O}_X(aD+bE)\big)
\end{tikzcd}$$

If $a,b\geqslant m_0$, the right vertical map is surjective by (1), and the upper horizontal map is surjective by (2). So the lower horizontal multiplication map is surjective for $a,b\geqslant m_0$. This completes the proof. □

Remark 6.1. In the case X being smooth the resolution (6.3) is actually finite by the Hilbert Syzygy Theorem.

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